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Review Article

FROM IIM TO AUGMENTED IIM: A POWERFUL TOOL FOR COMPLEX PROBLEMS USING CARTESIAN MESHES

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Abstract: The immersed interface method (IIM) first proposed in is an accurate numerical method for solving elliptic interface problems on Cartesian meshes. It is a sharp interface method that was intended to improve accuracy of the immersed boundary (IB) method. The IIM is second order accurate in the maximum norm (pointwise, strongest) while the IB method is first order accurate. The first IIM paper is one of the most downloaded one from the SIAM website and is one of the most cited papers. While IIM provided a way of accurate discretization of the partial differential equations (PDEs) with discontinuous coefficients, the augmented IIM first proposed in made the IIM much more efficient and faster by utilizing existing fast Poisson solvers. More important is that the augmented IIM provides an efficient way for multi-physics models with different governing equations, problems on irregular domains, multi-scales and multi-connected domains. A briefly introduction of the augmented strategy including some recently progress is presented in this article.

Keywords: Interface problem, augmented IIM, fast solvers, ADI method, discontinuous coefficient, multi-scale, jump conditions.

1. INTRODUCTION

Interface problems arise in many applications. For example, when there are two different materials, such as a fluid flow and a porous media, or the same material but at different states, such as water and ice, we are dealing with an interface problem. If ordinary or partial differential equations are used to model multi-phase flows, multi-physics problems, the parameters in the governing differential equations are typically discontinuous, or even the governing PDEs are different as in the Stokes and Darcy's coupling, and the source terms are often singular to reflect source/sink distributions along co-dimensional interfaces. Because of these irregularities, the solutions to the differential equations are typically non-smooth, or even discontinuous as in the example of the pressure inside and outside an inflated balloon. As a result, many standard numerical methods based on the assumption of smoothness of solutions do not work or work poorly for interface problems.

We use a parabolic (heat) equation to illustrate the problem,

$$u_t = (\beta u_x)_x + (\beta u_y)_y - f(\mathbf{x}, t) - \int_{\Gamma} v(\mathbf{X}(s), t) \delta(\mathbf{x} - \mathbf{X}(s)) ds, \quad (1)$$

$$\mathbf{x} \in \Omega = \Omega^+ \cup \Omega^-, \quad \mathbf{X}(s) \in \Gamma,$$

with some initial and boundary conditions, where Γ is an interface, see Figure 1 for an illustration. We assume that the coefficient β has a finite jump across the interface Γ corresponding to two different materials or states and with a source term along the interface as in Peskin's Immersed Boundary model (Peskin & McQueen, 1995). It is known that the IB method is a smoothing method and has shown to be first order accurate by this author and

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other scholars, see (Li, 2015; Mori, 2018).

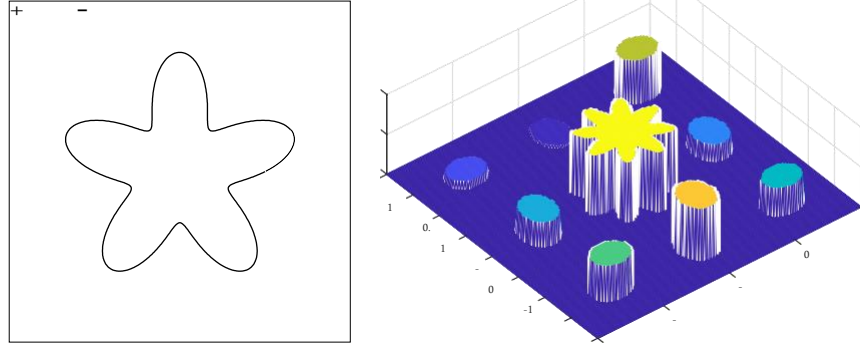


Figure 1: Left: a diagram of an arbitrary interface. Right: A diagram of a multiscale problem with multi-inclusions whose coefficient $\beta(x) \geq \beta_0 > 0$, which is a piecewise constant.

Using a sharp interface method that enforces the actually interface conditions instead of approximating the singular Dirac delta function δ , the above PDE can be written as

$$\begin{aligned} u_t &= (\beta u_x)_x + (\beta u_y)_y - f, \quad (x, y) \in \Omega^+ \cup \Omega^- \setminus \Gamma, \\ [u]_{\mathbf{x} \in \Gamma} &= 0, \quad \left[\beta \frac{\partial u}{\partial n} \right]_{\mathbf{x} \in \Gamma} = v(s, t), \end{aligned} \quad (2)$$

where the second jump is the flux jump and n is the normal direction with $|n|=1$ pointing to the Ω^+ side.

For a regular heat equation, the alternating direction implicit (ADI) method using a finite difference (FD) discretization is very efficient for time-dependent parabolic PDEs, see for example (Li, Qiao, & Tang, 2017). The classical ADI method is second-order accurate and unconditionally stable if the standard 3-point central finite difference discretization is used in each coordinate direction. An ADI method only requires to solve a sequence of tridiagonal systems of equations which can be done in $O(N^2)$ operations, and is considered as one of the most efficient algorithms.

The ADI method for singular sources ($v \neq 0$) but continuous coefficients ($[\beta]=0$) can be written as (Li & Mayo, 1993),

$$\begin{aligned} \frac{u_{ij}^{k+\frac{1}{2}} - u_{ij}^k}{\Delta t / 2} &= \delta_{xx}^h \beta u_{ij}^{k+\frac{1}{2}} - (C_x)_{ij}^{k+\frac{1}{2}} - Q_{ij}^k - R_{ij}^k + \delta_{yy}^h \beta u_{ij}^k - (C_y)_{ij}^k - f_{ij}^{k+\frac{1}{2}}, \\ \frac{u_{ij}^{k+1} - u_{ij}^{k+\frac{1}{2}}}{\Delta t / 2} &= \delta_{xx}^h \beta u_{ij}^{k+\frac{1}{2}} - (C_x)_{ij}^{k+\frac{1}{2}} - Q_{ij}^k - R_{ij}^k + \delta_{yy}^h \beta u_{ij}^{k+1} - (C_y)_{ij}^{k+1} - f_{ij}^{k+\frac{1}{2}}, \end{aligned} \quad (3)$$

where Δt is the time step size, $\delta_{yy}^h \beta u_{ij}^{k+1/2} - (C_x)_{ij}^{k+1/2}$ is a FD approximation to βu_{xx} at (x_i, y_i) at the time level $t^{k+1/2}$ and so on. At regular grid points, the correction terms $(C_x)_{i,j}^{k+1/2} = 0$ and

$$\delta_{xx}^h \beta u_{ij}^{k+1/2} = \frac{\beta_{i-1/2,j} U_{i-1,j}^{k+1/2} - (\beta_{i-1/2,j} + \beta_{i+1/2,j}) U_{ij}^{k+1/2} + \beta_{i+1/2,j} U_{i+1,j}^{k+1/2}}{h^2}. \quad (4)$$

When the coefficient is continuous, that is, $[\beta] = 0$, the irregularity is only from the right-hand side of the Dirac delta function corresponding to the flux jump in $[\beta \ u_n] = v$. In this case, we can apply the classical ADI finite difference scheme, but add non-zero correction terms at irregular grid points where the interface cuts through the standard central finite difference stencil. The key to an ADI method is to discretize the PDE dimension by dimension. For a heat equation with a constant coefficient, say $\beta = 1$, this can be and has been done easily since we know the jump $[u] = w$ (for more general problem) and the flux jump $[u_n] = v$, so the relations $[u]$, $[u_x]$, $[u_{xx}]$, and $[u]$, $[u_y]$, $[u_{yy}]$ from the problem and Γ directly, for example, with $\beta = 1$, we have

$$\begin{aligned} [u_x] &= [u_n] \cos \theta - [u]_\tau \sin \theta = v \cos \theta - \frac{dw}{d\tau} \sin \theta, \\ [u_y] &= [u_n] \sin \theta + [u]_\tau \cos \theta = v \sin \theta + \frac{dq}{d\tau} \cos \theta, \end{aligned} \quad (5)$$

where θ is the angle between the x-axis and the normal direction at the interface. The derivation of $[u_{xx}]$ and $[u_{yy}]$ can be found in (Li & Mayo, 1993). Assuming that the interface cuts the grid line $y = y_j$ at (x^*, y_j) , $x_i \leq x^* < x_{i+1}$, then the modified finite difference scheme for u_{xx} at (x_i, y_j) is

$$\frac{u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j))}{h^2} \approx u_{xx}(x_i, y_j) - (C_x)_{ij}, \quad (6)$$

where $h^2 (C_x)_{i,j} = \pm([u] + [u_x] + (x_{i+1} - x^*)^2 [u_{xx}]) / 2$. The correction term depends on known quantities w , v , and Γ linearly and is similar to the 'Taylor expansion' in terms of the jump conditions. Thus, the ADI method from time t^k to t^{k+1} can be written in the matrix-vector form as

$$A \mathbf{U}^{k+1} = \mathbf{F}_1^k - B_1 [\mathbf{U}]^{k*} - B_2 [\mathbf{U}_n]^{k*}, \quad (7)$$

where A is the classical ADI operator, $\mathbf{U}^{k+1}$ is the finite difference solution at t^{k+1} , typically $k^* = k+1/2$ is a time level between k and $k+1$, and $B_1 [\mathbf{U}]^{k*}$ and $B_2 [\mathbf{U}_n]^{k*}$ correspond to the correction terms resulted from $[u]$ and $[u_n]$, respectively. Note that the $\mathbf{U}^k$ term has been absorbed into $\mathbf{F}^k$, and the dimension of $[\mathbf{U}]^{k*}$ and $[\mathbf{U}_n]^{k*}$ are $O(N)$ compared with that of $\mathbf{U}^{k+1}$ which is $O(N^2)$.

However, if β has a jump across the interface Γ , then we do not have the dimensional jump relations anymore. The augmented strategy is to introduce one or several intermediate variables so that we can still use the ADI method. For example, we can set $\mathbf{Q}^{k*} = [\mathbf{U}_n]^{k*}$ as an unknown to have

$$A \mathbf{U}^{k+1} + B_2 \mathbf{Q}^{k*} = \mathbf{F}_1^k - B_1 [\mathbf{U}]^{k*}, \quad (8)$$

where $\mathbf{Q}^{k*}$ is an approximation to $[\mathbf{U}_n]^{k*}$ which should be chosen to satisfy the interface condition $[\beta \ u_n] = v$. In discretization, often a local interpolation is used to approximate $[\beta \ u_n] = v$ which can be written as another matrix-vector form

$$C \mathbf{U}^{k+1} + D \mathbf{Q}^{k*} = \mathbf{F}_2^k \quad (9)$$

for some sparse matrices C and D . The solution $(\mathbf{U}^{k+1}, \mathbf{Q}^{k*})$ satisfies the following system of equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{U}^{k+1} \\ \mathbf{Q}^{k*} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1^k \\ \mathbf{F}_2^k \end{bmatrix}. \quad (10)$$

The above system of equations provides a discretization process using the ADI method.

Another key of the augmented method is how to solve the above system of equations efficiently. Note that the Schur complement for $\mathbf{Q}^{k*}$ is

$$(D - CA^{-1}B)\mathbf{Q}^{k*} = \mathbf{F}_2^k - CA^{-1}\mathbf{F}_1^k = \bar{\mathbf{F}}^k, \quad \text{or} \quad S\mathbf{Q}^{k*} = \bar{\mathbf{F}}^k, \quad (11)$$

where $A^{-1}\mathbf{F}^k$ is the result of one step of the ADI for a regular problem. Often, we can solve the Schur complement system efficiently since the dimension of the unknown is $O(N)$. For fixed interface and time step size, we just need to form S and perform the LU decomposition once. For time dependent problems and adaptive time step size, the GMRES method is recommended.

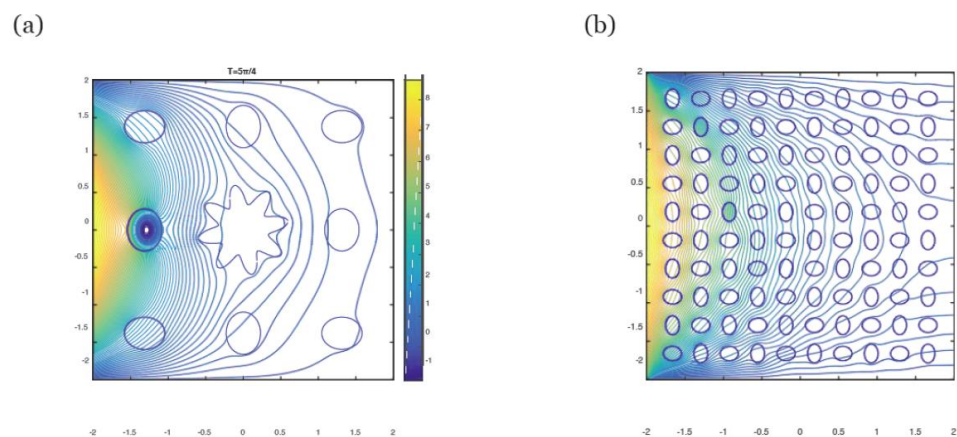


Figure 2: Snap contour plots of the computed solution with $N = 160$ of a flow through a porous media.

In Figure 2, we show plots of the streamlines of the flow from left passes different objects with different permeability. For particles with large permeability, the flow gets saturated while the flow goes around if the permeability is very small.

2. A BRIEF REVIEW OF AUGMENTED IIM

There are at least two motivations for using augmented strategies. The first one is to get a faster algorithm compared to a direct discretization, particularly to take advantage of existing fast solvers. The second one (more important one) is that, for some interface problems, an augmented approach may be the only way to derive an accurate algorithm. The augmented techniques enable us to decouple the jump conditions so that the idea of the immersed interface method can be applied, see for example, (Li, 2015; Li, Ito, & Lai, 2007; Li & Qin, 2017).

The application of the augmented approach for two dimensional interior irregular domains can be found in (Li, Zhao, & Gao, 1999) for simulating electro-migration of voids in an integrated circuit; in (Hunter, Li, & Zhao, 2002) for simulating the motion of drops; in (Li & Wang, 2003) for solving Navier-Stokes equations on irregular domains using the stream- vorticity formulation; in (Jin & Wang, 2002) for micro-structure evolution in the chemical vapor infiltration process, and (Chen, Li, & Lin, 2008) for solving bi-harmonic equations on irregular

domains; and recently work for bi-harmonic interface problems (Li & Qin, 2017). The augmented approach for generalized Helmholtz equations has also been developed in three dimensions and has been applied to an inverse problem of shape optimization in (Deng, Ito, & Li, 2003; Ito, Kunisch, & Li, 2001). While augmented methods have some similarities to boundary integral methods or integral equation approaches to find a source strength, the augmented methods have a few special features: (1) no Green function is needed, and therefore there is no need to evaluate singular integrals; (2) there is no need to set up the system of equations for the augmented variable explicitly; (3) they are applicable to general PDEs with or without source terms; (4) the process does not depend on the boundary conditions. On the other hand, we may have less knowledge about the condition number of the Schur complement system and how to apply pre-conditioning techniques.

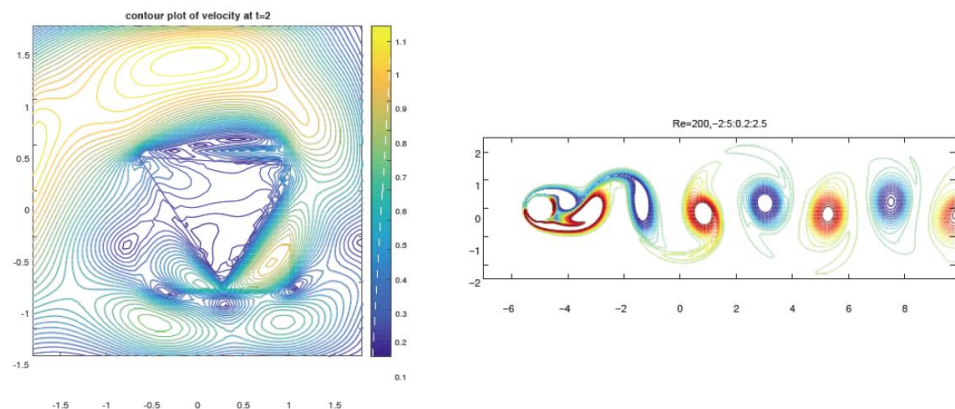


Figure 3: Simulations with the augmented IIM. (a): An interface with corners in a cavity flow, see (Li, Lai, Peng, & Zhang, 2018). (b): An irregular domain problem: a flow passes through a stationary cylinder computed with $Re = 200$, see (Ito, Lai, & Li, 2009).

Recent a remarkable progress on augmented IIM is the convergence proof and for variable coefficients. It has been shown in (Li, Ji, & Chen, 2016) that not only the computed solution is second order accurate globally in the strongest maximum norm, but also the computed gradient from each side of the interface. The method is designed for closed smooth interfaces not for open-ended interface such as cracks. For closed interfaces but with corners, see the left plot in Figure 3. The convergence of the augmented method for elliptic interface problems has been shown both in one and two dimensions under appropriate regularity assumptions and a piecewise constant $\beta(\mathbf{x})$. For a variable coefficient $\beta(\mathbf{x})$, the conclusions are still true if h is small enough, that is, in the asymptotic sense.

New applications of the augmented IIM include multi-physics simulations with different governing equations on different regions such as the Stokes or Navier-Stokes equations coupled with the Darcy's law in (Li, 2015; Li, Lai, Peng, & Zhang, 2018), determine the crank tips for a Mumford-Shah minimizer of a free boundary value problem in (Li & Mikayelyan, 2016); the 3D compressible bubbles in a incompressible fluid (Li, Xiao, Cai, Zhao, & Luo, 2015); and an efficient preconditioner for the Schur complement matrix resulted from AIIM in (Angot & Li, 2017; Xia, Li, & Ye, 2015).

REFERENCES

- Angot, P., & Li, Z. L. (2017). An augmented IIM & preconditioning technique for jump embedded boundary conditions. *International Journal of Numerical Analysis and Modeling*, 14(4-5), 712-729.
- Chen, G., Li, Z. L., & Lin, P. (2008). A fast finite difference method for biharmonic equations

- on irregular domains and its application to an incompressible Stokes flow. *Advances in Computational Mathematics*, 29(2), 113-133.
- Deng, S. Z., Ito, K., & Li, Z. L. (2003). Three-dimensional elliptic solvers for interface problems and applications. *Journal of Computational Physics*, 184(1), 215-243.
- Hunter, J. K., Li, Z. L., & Zhao, H. K. (2002). Reactive autophobic spreading of drops. *Journal of Computational Physics*, 183(2), 335-366.
- Ito, K., Kunisch, K., & Li, Z. L. (2001). Level-set function approach to an inverse interface problem. *Inverse Problems*, 17(5), 1225-1242.
- Ito, K., Lai, M. C., & Li, Z. (2009). A well-conditioned augmented system for solving Navier-Stokes equations in irregular domains. *Journal of Computational Physics*, 228(7), 2616-2628.
- Jin, S. & Wang, X. (2002). Robust numerical simulation of porosity evolution in chemical vapor infiltration, *Journal of Computational Physics*, 179, 557-577.
- Li, Z. L. (2016). An augmented Cartesian grid method for Stokes-Darcy fluid-structure interactions. *International Journal for Numerical Methods in Engineering*, 106(7), 556-575.
- Li, Z. L. (2015). On convergence of the immersed boundary method for elliptic interface problems, *Math. Comp.* 84 (293), 1169-1188.
- Z. Li, K. Ito, and M-C. Lai, An augmented approach for Stokes equations with a discontinuous viscosity and singular forces, *Computers and Fluids*, **36** (2007), 622-635.
- Li, Z. L., Ji, H. F., & Chen, X. H. (2017). Accurate solution and gradient computation for elliptic interface problems with variable coefficients. *SIAM Journal on Numerical Analysis*, 55(2), 570-597.
- Li, Z. L., Lai, M. C., Peng, X. F., & Zhang, Z. Y. (2018). A least squares augmented immersed interface method for solving Navier-Stokes and Darcy coupling equations. *Computers & Fluids*, 167, 384-399.
- Li, Z. L. & A. Mayo (1993), ADI methods for heat equations with discontinuities along an arbitrary interface, *AMS Proc. Symp. Appl. Math. W. Gautschi, editor*, 48, 311-315.
- Li, Z. L., & Mikayelyan, H. (2016). Fine numerical analysis of the crack-tip position for a Mumford-Shah minimizer. *Interfaces and Free Boundaries*, 18(1), 75-90.
- Li, Z., Qiao, Z., & Tang, T. (2018). *Numerical solution of differential equations: Introduction to finite difference and finite element methods*. New York: Cambridge University Press.
- Li, Z. L., & Qin, F. F. (2017). An augmented method for 4th order PDEs with discontinuous coefficients. *Journal of Scientific Computing*, 73(2-3), 968-979.
- Li, Z. L., & Wang, C. (2003), A fast finite difference method for solving Navier-Stokes equations on irregular domains, *J. of Commu. in Math. Sci.*, **1**, 180-196.
- Li, Z. L., Xiao, L., Cai, Q., Zhao, H. K., & Luo, R. (2015). A semi-implicit augmented IIM for Navier-Stokes equations with open, traction, or free boundary conditions. *Journal of Computational Physics*, 297, 182-193.
- Li, Z. L., Zhao, H. K., & Gao, H. J. (1999). A numerical study of electro-migration voiding by evolving level set functions on a fixed Cartesian grid. *Journal of Computational Physics*, 152(1), 281-304.
- Mori, Y. (2008) Convergence proof of the velocity field for a Stokes flow immersed boundary method, *Comm. Pure Appl. Math.* **61**, 1213-1263.
- Peskin, C. & McQueen, D. M. (1995), A general method for the computer simulation of

biological systems interacting with fluids, *Symposia of the Society for Experimental Biology* **49**, 265.

Xia, J. L., Li, Z. L., & Ye, X. (2015). Effective matrix-free preconditioning for the augmented immersed interface method. *Journal of Computational Physics*, 303, 295-312.



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