



Original Research Article

A WEIGHTED BINOMIAL LINDLEY DISTRIBUTION FOR DISCRETE DATA

Tassaddaq Hussain

Department of Mathematics, Mirpur University of Science and Technology (MUST), Mirpur 10250 Azad Jammu & Kashmir Pakistan.

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Correspondance:

Tassaddaq Hussain PhD,
Department of Mathematics, Mirpur
University of Science and Technology
(MUST), Allama Iqbal Road, Mirpur
Azad Jammu & Kashmir 10250 Mirpur
Pakistan. C.: +92 333 8264078 | Tel.:
+92-5827-961040 / 42 | F.: +92-5827-
961040 / 42 | E.:
tafkho2000@gmail.com /
info@must.edu.pk |
<https://must.edu.pk/>

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Abstract

In this article we propose a weighted and mixed version of binomial and discrete Lindley distributions. The weighted distributions are usually considered for the adjustment of probability of occurrences and removal of biasedness from the data and mixed distributions are usually used in heterogeneous and over-dispersed data sets. The proposed model adjusts the probabilities by removing the bias but contrary to the over-dispersion the model handles the under dispersion issues. Moreover, various properties like failure rate, moments, and factorial moments, recurrence relation between moments, self-decomposability, infinite divisibility and limiting form of the proposed model are also studied. A simulation is conducted and estimation problem is discussed via two estimation methods like maximum likelihood and moments estimation. Performance of the proposed model over the binomial, discrete gamma, Poisson-Lindley and discrete Lindley is investigated via some test statistics and two under dispersed real data sets. We introduce a two parameter discrete distribution, so-called the WBL distribution. The new distribution is much more flexible than the binomial, discrete Lindley distribution and could have increasing and unimodal hazard rates. We derive explicit algebraic formula for the r^{th} moment which holds in generality for any parameter values. The estimation procedures for the parameters of WBL distribution are also derived considering two estimation methods whose convergence rate is very high as compare to the others. The MME method yields global estimates where as MLE yields local estimates. Since it is not feasible to compare these methods theoretically, we have presented an extensive simulation study in order to identify the most efficient procedure. We observed that the MLE provides a large number of local estimates whereas MME yields global one so we used this global value as a seed for finding the MLE. Moreover, we have also studied various mathematical properties including moments, ascending and descending factorial moments, negative and negative factorial moments, discrete self-decomposability, and infinite divisibility. Finally, two data-sets were analyzed for illustrative purposes

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1. Introduction

The statement of Patil and Rao (1977) is worth stating before introducing the goal of this paper, according to them; Although the situations that involve weighted distributions seem to occur frequently in various fields, the underlying concept of weighted distributions as a major stochastic concept does not seem to have been widely recognized. This quote has become essential milestone for the possible solution of modern era problems but work on this topic is still rare. As a matter of fact, it is frequently seen that researchers can neither select sampling units in observational studies with equal probability nor have access to well defined sampling frames, e.g., for human, wildlife, insect, plant or fish populations. Moreover, the recorded observations on individuals among the above mentioned populations are biased and not have the original distributions unless every observation is given an equal chance of being recorded. Since, biased data arise in all disciplines of science, researchers have investigated adhoc solutions for correction of the biases. In this context, Rao (1985) and Patil (1991) studied the weighted distribution in a unified manner for modeling the biased data which is not based on complete sampling frames and units are not selected with equal probabilities. Such situations are more likely to be observed in non-experimental, nonreplicated and non-random categories, which are due to one or more reason viz. non-observability of some events, damage caused to original observations and adoption of unequal probability sampling.

For removing bias and arriving to the correct distribution, researchers usually adopt the approach of weighted concept of biased observation which leads towards the development of weighted distribution. A formal definition of a weighted distribution is obtained by considering a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $X: \Omega \rightarrow \mathcal{R}$, then at non-negative integer valued random variable its probability mass function can be defined as $\mathbb{P}(X = x) = p(k; \theta)$, $x \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ depending on parameter $\theta \in \Theta \subseteq \mathbb{R}^d$ and let $\omega(x)$ be a non negative function on $\mathbb{N}_0$ which need not to be zero and one, may exceed unity but should have a finite expectation i.e., $\mathbb{E}(\omega(X)) = \sum_{x \in \text{event}} \omega(x) \mathbb{P}(X = x) < \infty$.

Suppose that when the event occurs, the weight $\omega(k)$ is used to adjust the probability. Therefore, the recorded x is thus the realization of count random variable Y which is the weighted version of X . The recorded x is thus the realization of count random variable Y which is the weighted version of X . Then its probability mass function is of the form

(1)

$$\mathbb{P}(Y = x) = p_Y(x; \theta) = \frac{w(x; \psi) \mathbb{P}(X = x)}{\mathbb{E}(w(X; \phi))}, x \in \mathbb{N}_0,$$

where the discrete function $\omega(k) \equiv \omega(k; \psi)$ can depend on a parameter ψ representing the recording mechanism, and it may also be connected to the underlying initial parameter θ . More details on weight functions are available in Patil and Rao (1986). Some particular cases of weight functions are the standard distributions with $\omega(x; \psi) = C \forall x \in \mathbb{N}_0$, the size-biased distributions with weight $\omega(x; \psi) = x \forall x \in \mathbb{N}_0$, the popular COM-Poisson with weight $\omega(x; \psi) = (x!)^{1-\psi} \forall x \in \mathbb{N}_0$ (see Balakrishnan, 2014, pp: 511-512), Neel and Schull (1966) proposed the weighted Poisson distribution with weight $\omega(x; \psi) = \frac{\psi^k e^{-\psi}}{x!}$, $\psi > 0 \forall x \in \mathbb{N}_0$ and weighted binomial distribution with weight function $\omega(x; \psi) = 1 - (1 - \beta)^x$, $0 < \beta < 1, \forall x \in \mathbb{N}_0$ (see Kokonendji and Casany, 2012). For a detail count of weight functions readers refer to Johnson et al. (2005).

1.1. Gensis and Definition of Distribution:

Hussain et al.(2016)introduced a discrete Lindley distribution with two parameters Lindley distribution having the following probability mass function (pmf)

$$\mathbb{P}(X = x) = p_x = \frac{(1 - p)^2(1 + \beta x)p^x}{(1 + \beta p - p)}, \quad (2)$$

$$k = 0, 1, 2, 3, \dots, 0 < p < 1 \text{ and } \beta \in \mathbb{Z}^+.$$

The above model is a mixture of geometric distribution ($Geo(p)$) and size biased Negative Binomial distribution ($SBNB(1; p)$) with proportions $\frac{1-p}{1-p+\beta p}$ and $\frac{\beta p}{1-p+\beta p}$, respectively.

Definition: Let X be a discrete random variable with probability function $\mathbb{P}(X = x)$. Suppose that X denotes the number of individuals in a group or a colony and $\mathbb{P}(X = x)$ is the probability that a group consists of x individuals. Let us suppose that a group is recorded only when at least one of the individuals in the group; out of β number of individuals which are linearly increases on each trial, is sighted and each individual has an independent chance p of being sighted and $1 - p$ of being not sighted. Then the probability that an observed group having x individuals is obtained by introducing a new weight function $\omega(x; \beta) = \binom{\beta}{x} (1 - p)^{\beta-x}$ and substituting it along with equation (2) into equation (1), thus we get

$$\mathbb{P}(Y = x) = p_x = \frac{1}{1 + \beta^2 p} \binom{\beta}{x} (1 + \beta x)p^x(1 - p)^{\beta-x}, \quad (3)$$

$$x = 0, 1, 2, 3, \dots, \beta, 0 < p < 1, q = 1 - p \text{ and } \beta \in \mathbb{Z}^+.$$

It is also noticed that model (3) is recognized as the mixture of binomial(B) distribution and size biased binomial distribution (SBB), that is

$$p_x = \frac{1}{1 + p\beta^2} B(x; \beta, p) + \frac{p\beta^2}{1 + p\beta^2} SBB(x - 1; \beta, p),$$

and weighted version of binomial (B) distribution with weight $\omega(x; \beta) = 1 + \beta x$ so known as weighted binomial Lindley distribution ($\mathcal{WBL}$)

Now, we show the genesis of the $\mathcal{WBL}$ model using the following motivations

Motivation i): The proposed model $\mathcal{WBL}$ yields when the trials are dichotomous, independent, with a constant probability of success, repeated a fixed number of times say β and number of successes increases linearly on each trial. Motivation ii): The proposed model is unimodal, as its pmf satisfies the inequality $(p_x)^2 \geq p_{x-1}p_{x+1}$, for all x , i.e. $1 + \beta\{(1 + \beta x)^2 + 2x + \beta^2 x\} \geq 0$, which is an important implications on the characteristic of its reliability function, failure rate function, tail probabilities and moments. The log-concave distributions produce an increasing failure rate and a monotonically decreasing

mean residual life time function.

Motivation iii): $\mathcal{WBL}$ is an infinitely divisible and self decomposable.

Motivation iv): Although the mixtures work very well in heterogeneity and over dispersion yet $\mathcal{WBL}$ works very well in situation where less variation is observed in the data than predicted i.e. under dispersion.

Further motivations for the $\mathcal{WBL}$ distribution are given in the following notes.

Models with under-dispersion phenomena are rarely studied in the statistical literature. However, there are probability models which possess over- and under-dispersion. Therefore, our probability model only explains the circumstances where one observes less variation in the data.

Other motivations for the $\mathcal{WBL}$ distribution are stated in the following. As $\mathcal{WBL}$ having two

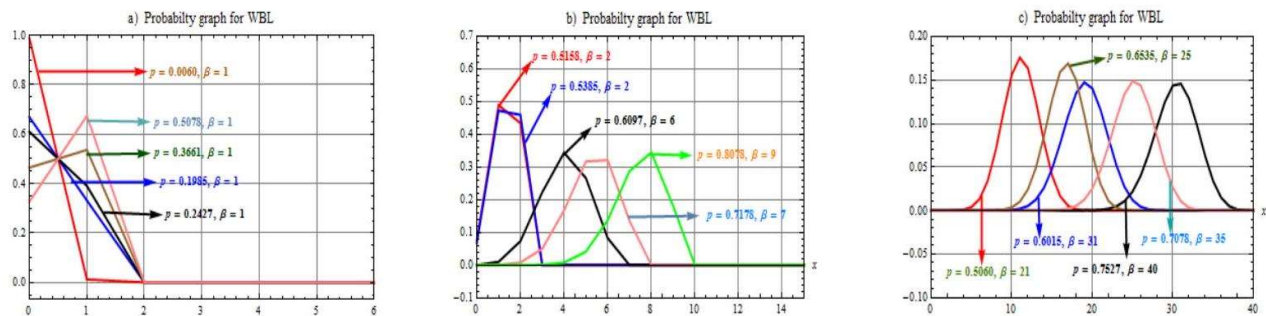


Figure 1: $\mathcal{WBL}$ probability plots for the indicated Values

parameters, namely shape parameter p and scale parameter β , so its behavior in the presence of these parameters is valuable and summarized in the following notes. The model shows different curve behavior which varies from symmetry to reverse J shapes. For smaller values of the shape parameter i.e., $p \rightarrow 0$, the distribution has reverse J shapes until $\beta < 1$. However, as β increases the i.e., $\beta > 1$, mode of the distribution moves towards right with the increase of shape parameter p . Figure 1 confirms the above statement. Moreover, the recursive relation between $\mathcal{WBL}$ probabilities can be expressed as

(4)

$$(1 + \beta x)(x + 1)P(Y = x + 1)(1 - p) = p(1 + \beta(x + 1))(\beta - x)P(Y = x),$$

The failure, survival and hazard rate functions of the $\mathcal{WBL}$ distribution are, respectively, given as

(5)

$$\Psi_x = \frac{1}{1 + \beta^2 p} \{ \mathbb{I}_{1-p}(\beta - x, x + 1) + \beta^2 p \mathbb{I}_{1-p}(\beta - x - 1, x) \},$$

$$\varphi_x = 1 - \Psi_x = \frac{1}{1 + \beta^2 p} \{ \mathbb{I}_p(x + 1, \beta - x) + \beta^2 p \mathbb{I}_p(x, \beta - x - 1) \},$$

And

(6)

$$\hbar_x = \frac{1 + \beta^2 p}{\mathbb{I}_p(x + 1, \beta - x) + \beta^2 p \mathbb{I}_p(x, \beta - x - 1)},$$

where $\mathbb{I}_x(a, b) = \int_0^x u^{a-1} (1-u)^{b-1} du = \frac{\mathbb{I}(x; a, b)}{\mathbb{I}(a, b)}$, $\mathbb{I}(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} du$ and ${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}$ denote the incomplete beta, complete beta and hyper geometric series functions respectively. Also, the incomplete beta function is the cumulative distribution function of the Beta

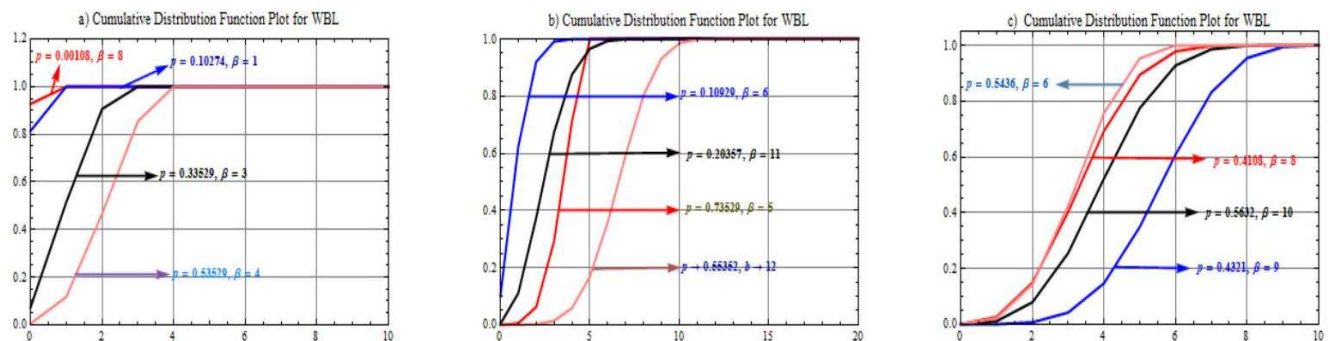


Figure 2: WBL CDF graphs for the indicated values

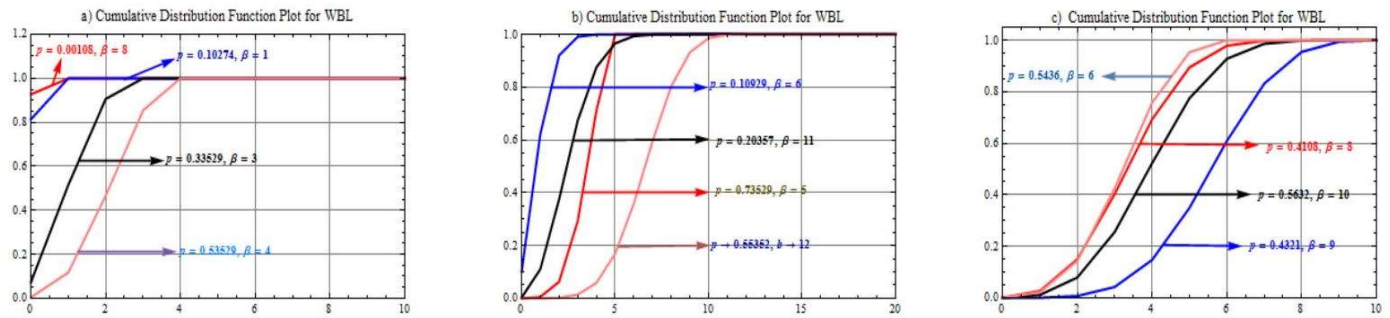


Figure 2: WBL CDF graphs for the indicated values

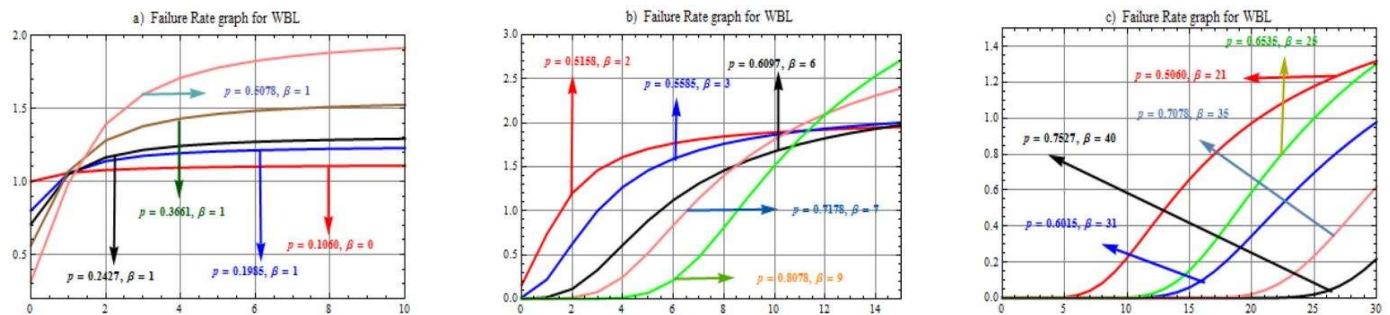


Figure 3: WBL Failure Rate graphs for the indicated values

distribution, and is related to the cumulative distribution function of a random variable X from a binomial distribution, where the "probability of success" is p and the sample size is n : $\varphi(x; n, p) = \mathbb{P}(X \leq x) = \sum_{k=0}^x \binom{n}{k} p^k (1-p)^{n-k} = \mathbb{I}_{1-p}(n-x, x+1) = \mathbb{I}_p(x+1, n-x)$. Figure 2 gives the cumulative distribution function (cdf) plots of WBL and indicates the convergence pattern of the distribution, for the plots it is observed that for smaller values of p and for all values of β , the cdf takes higher values and then converge to 1. However, as p and for all values of β increase, the cdf takes initiate from smaller values and then converge to 1. Clearly,

$$\lim_{x \rightarrow \infty} h_x = \frac{1}{1-p},$$

therefore, the hazard rate function of the WBL distribution is bounded above and have increasing failure rate (IFR) which is an important property for the lifetime models. The curve behavior of the hazard rate can also be visualized from Figure 3.

Further, the mean residual life function of the WBL is defined by

$$\mathbb{E}(Y - x \mid Y \geq x) = \frac{\sum_{k=x}^{\infty} k P(Y = k)}{\sum_{k=x}^{\infty} P(Y = k)} - x,$$

$$\text{MRL}(Y) = \frac{(\beta-x)}{h_x} \left\{ (1+\beta x)_2 F_1 \left(-\beta + x + 1, 2; x + 2; -\frac{1}{1-p} \right) + \beta_3 F_2 \left(-\beta + x + 1, 2, 2; x + 2, 1; -\frac{1}{1-p} \right) \right\}.$$

1.2. Outline for rest the paper:

Rest of the paper is organized as follows. Section 2 gives several mathematical properties of the $\mathcal{WBL}$ distribution, such as, probability generating function, moments, factorial moments, recurrence relation between moments, negative moments, self decomposability and infinite divisibility. In Section 3 estimation of the parameters via maximum likelihood method with associated Fisher information matrix, method of moments and simulation are studied. Test and evaluation statistics as well as three

real data application for the $\mathcal{WBL}$ distribution are discussed in Section 4. Finally, Section 5 deals

with the drawn conclusion.

2. Some mathematical mproperties of the WBL distribution

The recognition of any discrete probability distribution is usually based on its probability generat-

ing function (pgf). The probability generating function (pgf) of the random variable $Y \sim \mathcal{WBL}(p, \beta)$

is defined and expressed as $G_Y(t) = \mathbb{E}(t^X) = \sum_{x=0}^{\beta} t^x \mathbb{P}(Y = x)$,

(7)

$$G_Y(t) = \frac{(1 - p + pt)^{\beta-1}(1 - p + pt + \beta^2 pt)}{1 + \beta^2 p},$$

where $\beta \geq 0, 0 < p < 1$ for $0 < pt < 1$.

If t is replaced by e^t , we get the moment generating function (mgf) of the $\mathcal{WBL}$ distribution as

(8)

$$M_Y(t) = \frac{(1 - p + pe^t)^{\beta-1}(1 - p + pe^t + \beta^2 pe^t)}{1 + \beta^2 p},$$

where $0 < pe^t < 1$.

The r^{th} derivatives of equation (8) with respect to t at $t = 0$ yield the moments, particularly, for $r = 1$ we have

for $r = 2$ we have

$$\mu'_1 = \frac{\beta p(1 + \beta^2 p + \beta(1 - p))}{1 + \beta^2 p},$$

$$\mu'_2 = \frac{(\beta p(1 + \beta + (\beta - 1)(1 + 3\beta)p + (\beta - 2)(\beta - 1)\beta p^2))}{1 + \beta^2 p},$$

for $r = 3$ we have

$$\mu'_3 = \frac{\beta p(1 + \beta + (\beta - 1)(3 + 7\beta)p + (\beta - 2)(\beta - 1)(1 + 6\beta)p^2 + (\beta - 3)(\beta - 2)(\beta - 1)\beta p^3)}{1 + \beta^2 p},$$

and for $r = 4$ we have

$$\mu'_4 = \frac{p(\beta - 1)(1 + p(\beta - 2))(7 + p(\beta - 3))(6 + p(\beta - 4)) + p(\beta^2 + 1)(1 + 2p(\beta - 1))(7 + (\beta - 2)p)(9 + 2p(\beta - 3))}{1 + \beta^2 p}.$$

Now, the r^{th} moment about origin can also be expressed as

$$\mu'_r = \frac{1}{1 + \beta^2 p} \sum_{x=0}^{\beta} x^r \binom{\beta}{x} (1 + \beta x)p^x (1 - p)^{\beta-x},$$

on differentiating the above equation with respect to p we get

$$\begin{aligned} \frac{d\mu'_r}{dp} = & - \sum_{x=0}^{\beta} x^r \binom{\beta}{x} (1 + \beta x)p^x \left\{ \frac{\beta^2(1 - p)^{\beta-x}}{(1 + \beta^2 p)^2} + \frac{(\beta - x)(1 - p)^{\beta-x-1}}{(1 + \beta^2 p)} \right\} \\ & + \frac{1}{(1 + \beta^2 p)p} \sum_{x=0}^{\infty} x^{r+1} \binom{\beta}{x} (1 + \beta x)p^x (1 - p)^{\beta-x}, \end{aligned}$$

which on simplification yields the recursive relation

$$\mu'_{r+1} = p(1 - p) \frac{d\mu'_r}{dp} + \frac{\beta p(1 + \beta^2 p + \beta(1 - p))}{1 + \beta^2 p} \mu'_r.$$

$r = 0, 1, 2, 3, \dots, \mu'_0 = 1, 0 < p < 1, q = 1 - p \text{ and } \beta \in \mathbb{Z}^+.$

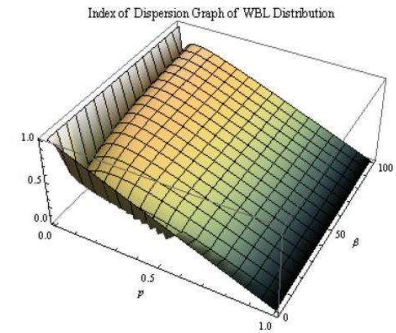
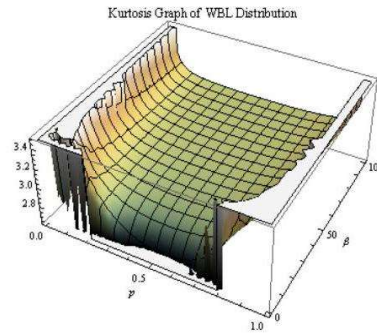
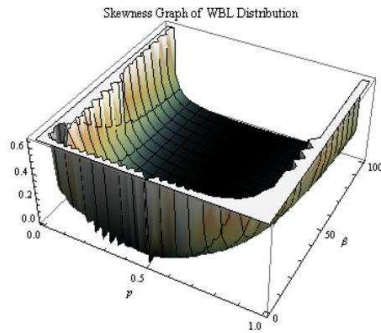
Similarly, by definition the r^{th} central moment can be expressed as

$$\mu_r = \frac{1}{1 + \beta^2 p} \sum_{x=0}^{\beta} \left(x - \frac{\beta p(1 + \beta^2 p + \beta(1 - p))}{1 + \beta^2 p} \right)^r \binom{\beta}{x} (1 + \beta x)p^x (1 - p)^{\beta-x},$$

now on differentiating the above equation with respect to p and simplifying it we get

$$\begin{aligned}\mu_{r+1} + \mu'_1 \mu_r &= p(1-p) \frac{d\mu_r}{dp} + \frac{p\beta(1 + \beta^2 p + \beta(1-p))}{1 + \beta^2 p} \mu_r \\ &+ \frac{rp(1-p)(1 + \beta^2 + (\beta-1)(1 + \beta^2 p))}{1 + \beta^2 p} \mu_{r-1}.\end{aligned}$$

$r = 1, 2, 3, \dots, \mu'_0 = 1, \mu_1 = 0, 0 < p < 1, q = 1 - p \text{ and } \beta \in \mathbb{Z}^+.$



Index of Dispersion: In statistics, the phenomena of over- and under-dispersion relative to the Poisson distribution are generally observed in count data and well known in statistical literature. There are various causes of such phenomenon, like heterogeneity and aggregation for over-dispersion and repulsion for under-dispersion although less frequent (see Kokonendji and Mizre, 2005). In order to see the reflection of over- and under-dispersion pattern researchers usually take the support of the Index of dispersion (ID) which is defined as the ratio of variance to mean. If $ID > 1$ (< 1) the distribution is over-dispersed (under-dispersed). The index of dispersion for the WBL distribution is expressed as

$$ID = \frac{(1-p)(1 + \beta + 2\beta(\beta-1)p + (\beta-1)\beta^3 p^2)}{((1 + \beta + (\beta-1)\beta p)(1 + \beta^2 p))}.$$

As $p \rightarrow 0$ the distribution is equi-dispersed and it is under-dispersed for all values of β as $p \rightarrow 1$.

Similarly, another important statistical properties is the skewness which usually refer to the departure from symmetry. In order to measure such departure we define co-efficient of skewness which is defined as the ratio of cube of third mean moment to the square of the second mean moment, for the WBL distribution it is expressed as

$$\begin{aligned}SK. &= \frac{1}{(\beta(1-p)p(1 + \beta + 2(\beta-1)\beta p + (\beta-1)\beta^3 p^2)^3)} \times \\ &\{(1 + \beta - (2 + \beta(6 + (\beta-3)\beta))p + 3(\beta-1)\beta(\beta^2 - 2)p^2 \\ &+ (\beta-1)\beta^3(\beta^2 - 6)p^3 - 2(\beta-1)\beta^5 p^4)^2\}.\end{aligned}$$

The distribution is symmetrical in the interval $0.41 < p < 0.59$ the interval increases as β increases. Kurtosis is another statistical measure that measures the degree of peakedness or flatness of a unimodal frequency curve, it is the ratio of the fourth mean moment to the square of second mean moment, for WBL it is expressed as

$$\text{KUR.} = \frac{1}{(\beta(1-p)p(1+\beta+2(\beta-1)\beta p+(\beta-1)\beta^3 p^2)^2)} \times \\ \{(1+\beta-(6+\beta(11+(-10+\beta)\beta))p \\ +(6+\beta(33+\beta(-42+\beta(-2+\beta(18+\beta))))p^2 \\ +2(-1+\beta)\beta(12+\beta(-6+\beta(-30+\beta(9+5\beta))))p^3 \\ +(-1+\beta)\beta^3(48+\beta(-18+\beta(-42+\beta(12+\beta))))p^4 \\ +3(-3+\beta)(-2+\beta)(-1+\beta)\beta^5(2+\beta)p^5-3(-3+\beta)(-1+\beta)\beta^7 p^6)\}.$$

The distribution leptokurtic for $p < 0.20$ and β increases, mesokurtic for $p = 0.20$ and $p = 0.80$ and for $0.20 \leq p \leq 0.80$ it has platykurtic behavior. It is also observed that the distribution is platykurtic for $\beta \rightarrow 0$ until $p < 0.80$, however if $p \rightarrow 1$ the distribution shows leptokurtic behavior.

2.1. Factorial Moments

The r^{th} descending factorial moment for a r.v. $Y \sim \mathcal{WBL}(p, \beta)$ can be defined as $\mu'_{(r)} = \mathbb{E}(Y^{(r)}) = \sum_{x=0}^{\beta} x^{(r)} \mathbb{P}(Y = x)$, now by using the expression $x^{(r)} = x(x-1) \dots (x-r+1) = \frac{x!}{(x-r)!}$, we have

$$\mu'_{(r)} = \frac{(1-p)^{\beta}}{(1+\beta^2 p)} \sum_{x=r}^{\beta} \frac{x!}{(x-r)!} \binom{\beta}{x} (1+\beta x) \left(\frac{p}{1-p}\right)^x.$$

By using the binomial series $(a+x)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k}$ and after some algebraic manipulation equation is attained to

(9)

$$\mu'_{(r)} = \frac{p^r r! (1+\beta r) \binom{\beta}{r} (r+1+\beta(\beta-r)p)}{(1+\beta^2 p)(r+1)},$$

which generates a recursive relation between r and $r+1$ descending factorial moments as

(10)

$$\mu'_{(r+1)}(1+\beta(\beta p+r-pr)) = \mu'_{(r)}(p(\beta-r)(1+\beta(1+p(\beta-r-1)+r))).$$

where $r = 0, 1, \dots$ and $\mu'_{(0)} = 1$.

The r^{th} , ascending factorial moment of $Y \sim \mathcal{WBL}(p, \beta)$ can be defined and expressed as $\mu'_{[r]} = \mathbb{E}((Y)_r) = \sum_{x=0}^{\beta} (x)_r \mathbb{P}(Y = x)$

$$\mu'_{[r]} = \frac{(1-p)^{\beta}}{1+p\beta^2} \left\{ \sum_{x=0}^{\beta} \binom{\beta}{x} (x)_r p^x + \beta \sum_{x=0}^{\beta} \binom{\beta}{x} x(x)_r p^x \right\},$$

after simplification on using the hyper geometric function ${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}$, we get

$$\mu'_{[r]} = \frac{(1-p)^\beta \beta pr!}{1+p\beta^2} \left\{ {}_2F_1\left(-\beta+1, r+1; 2; -\frac{p}{1-p}\right) + \beta {}_2F_1\left(-\beta+1, r+1; 1; -\frac{p}{1-p}\right) \right\},$$

where

$$\sum_{x=0}^{\beta} \binom{\beta}{x} (x)_r \left(\frac{p}{1-p}\right)^x = \beta pr! {}_2F_1\left(-\beta+1, r+1; 2; -\frac{p}{1-p}\right)$$

and

$$\beta \sum_{x=0}^{\beta} \binom{\beta}{x} x(x)_r \left(\frac{p}{1-p}\right)^x = \beta^2 r! {}_2F_1\left(-\beta+1, r+1; 1; -\frac{p}{1-p}\right).$$

where $r = 0, 1, 2, \dots, \mu'_{[0]} = 1 - \frac{(1-p)^\beta}{1+p\beta^2}$ and $(a)_n = a(a+1)(a+2) \dots (a+n-1)$.

2.2. Negative Moments

In the modern era of the inverse theory of random variables, the negative moments are gaining momentum in life testing phenomena, estimation purposes and identifying the models. The negative moments are being used in irreversible damage to manufacturing materials due to damage process such as creep, corrosion, creep fracture, wear, shrinkage, aging and cracking etc. (see Ahmad, 2007). These are also very appropriate for situations when the average of rates is desired. So, the first order moment of $\mathcal{WBL}$ is defined and expressed as $\mathbb{E}\left(\frac{1}{Y+a}\right) = \sum_{x=0}^{\beta} \frac{\mathbb{P}(Y=x)}{x+a}$, we have

$$\mathbb{E}\left(\frac{1}{Y+a}\right) = \frac{(1-p)^\beta}{1+p\beta^2} \sum_{x=0}^{\beta} \binom{\beta}{x} (x+a)^{-1} \left(\frac{p}{1-p}\right)^x (1+\beta x),$$

on simplification we get

$$\begin{aligned} \mathbb{E}(Y+a)^{-1} = & \frac{(1-p)^\beta}{1+p\beta^2} \left\{ \frac{1}{a} {}_2F_1\left(-\beta, a; a+1; -\frac{p}{1-p}\right) \right. \\ & \left. + \frac{\beta^2}{(a+1)(1-p)} {}_2F_1\left(-\beta+1, a+1; a+2; -\frac{p}{1-p}\right) \right\}, \end{aligned}$$

where

$$\sum_{x=0}^{\beta} \binom{\beta}{x} (x+a)^{-1} \left(\frac{p}{1-p}\right)^x = \frac{1}{a} {}_2F_1\left(-\beta, a; a+1; -\frac{p}{1-p}\right),$$

$$\sum_{x=0}^{\beta} \beta x \binom{\beta}{x} (x+a)^{-1} \left(\frac{p}{1-p}\right)^x = \frac{1}{(a+1)(1-p)} {}_2F_1\left(-\beta+1, a+1; a+2; -\frac{p}{1-p}\right),$$

$a > 0$.

Similarly, the s^{th} order negative moment of $Y \sim \mathcal{WBL}(p, \beta)$ can be expressed as

$$\begin{aligned} \mathbb{E}(Y+a)^{-s} &= \frac{(1-p)^{\beta}}{1+p\beta^2} \left\{ \frac{1}{a^s} s + 1 \right. \\ &\quad \left. + \frac{\beta^2 p}{(a+1)^s} s + 1 \right. \\ &\quad \left. {}_sF_s\left(a+1, \dots, a+1, -\beta+1; a+2, \dots, a+2; -\frac{p}{1-p}\right) \right\}, \end{aligned}$$

where

$${}_sF_u(a_1, \dots, a_s; b_1, \dots, b_u; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_s)_n z^n}{(b_1)_n \dots (b_u)_n n!},$$

the series converges for $s = u + 1$ and $|z| < 1$.

Negative factorial moment: The s^{th} order negative factorial moment of $Y \sim \mathcal{WBL}(p, \beta)$ is defined and expressed as

$$\mu'_{-[s]} = \mathbb{E}\left(\left(\frac{1}{Y+a}\right)^{[s]}\right) = \sum_{x=0}^{\beta} \prod_{i=0}^{s-1} (x+a+i)^{-1} \mathbb{P}(Y=x),$$

on incorporating equation (3) we get

$$\mu'_{-[s]} = \frac{(1-p)^{\beta}}{1+p\beta^2} \sum_{x=0}^{\beta} \prod_{i=0}^{s-1} (x+a+i)^{-1} \binom{\beta}{x} \left(\frac{p}{1-p}\right)^x (1+\beta x),$$

which after simplification yields

$$\begin{aligned} \mu'_{-[s]} &= \frac{(1-p)^{\beta}}{1+p\beta^2} \left\{ \frac{1}{(a)_s} {}_2F_1\left(a, -\beta; a+s; -\frac{p}{1-p}\right) \right. \\ &\quad \left. + \frac{\beta^2 p}{(a+1)_s} {}_2F_1\left(a+1, -\beta+1; a+s+1; -\frac{p}{1-p}\right) \right\}, \end{aligned}$$

where

$$\sum_{x=0}^{\beta} \prod_{i=0}^{s-1} \binom{\beta}{x} (x+a+i)^{-1} \left(\frac{p}{1-p}\right)^x = \frac{1}{(a)_s} {}_2F_1\left(a, -\beta; a+s; -\frac{p}{1-p}\right)$$

$$\sum_{x=0}^{\beta} \prod_{i=0}^{s-1} \binom{\beta}{x} (x+a+i)^{-1} \beta x \left(\frac{p}{1-p}\right)^x = \frac{\beta^2 p}{(a+1)_s} {}_2F_1\left(a+1, -\beta+1; a+s+1; -\frac{p}{1-p}\right).$$

where $a > 0$ and $s = 0, 1, 2, 3, \dots$

2.3. Discrete analogues of self-decomposability and infinite divisibility:

Definition (α -thinning): Let X be a non-negative integer-valued random variable and let $\alpha \in [0, 1]$, then the α -thinning of X using the Steutel and van Harn operator is defined by

$$\alpha \circ X \equiv Z_1 + Z_2 + \dots + Z_X$$

where $Z_i (i = 1, 2, \dots, X)$ are mutually independent Bernoulli random variables with $P(Z_i = 1) = \alpha$ and $P(Z_i = 0) = 1 - \alpha$. Additionally, the random variables Z_i are independent of X . The pgf $G_{\alpha \circ X}(s)$ of the random variable $\alpha \circ X$ is

$$G_{\alpha \circ X}(s) = G_X(1 - \alpha + \alpha s).$$

Definition(Discrete Infinite divisibility): A non-negative integer valued variable X is said to be infinitely-divisible if, for every non-negative integer n , the random variable X can be represented as a sum of n independent and identically distributed (i.i.d.) non-negative random variables $X_{n,i}; i = 1, 2, 3, \dots, n$; that is

$$X \equiv X_{n,1} + X_{n,2} + \dots + X_{n,n}, n = 1, 2, 3, \dots$$

(Note that the distributions of X and $X_{n,i} (i = 1, 2, 3, \dots, n)$ do not necessarily have to be of the same family.) Moreover, a random variable X is said to be discrete infinitely-divisible if the random variables $X_{n,i}$ only take non-negative integer values. A necessary condition for such a random variable X to be well defined is $P(X = 0) > 0$ (to exclude the trivial case we assume $(X = 0) < 1$) (see Steutel and Harn, 1979).

Definition (Discrete self-decomposability): A non-negative integer-valued random variable X is said to be discrete self-decomposable if for every $\alpha \in (0, 1)$ the random variable X can be written as $X \equiv \alpha \circ X + X_\alpha$ where random variables $\alpha \circ X$ and X_α are independent, which in terms of probability generating function can be expressed as

$$G_X(s) = G_X(1 - \alpha + \alpha s)G_{X_\alpha}(s),$$

where $G_{X_\alpha}(s)$ is the pgf of X_α . We have noted that the distribution of X_α uniquely determines the distribution of X . Moreover, a discrete infinitely-divisible distribution is discrete self-decomposable. By using the above definition the following proposition show that $\mathcal{WB}\mathcal{L}$ is a self decomposable and infinitely-

divisible distribution.

Proposition 1: If X_1 and X_2 are independent random variables such that $X_1 \sim \text{Bernoulli}\left(\frac{p(\beta^2+1)}{1+p\beta^2}\right)$ and $X_2 \sim \text{Binomial}(\beta-1, p)$, then $Y = X_1 + X_2$ has $\mathcal{WBL}(p, \beta)$ distribution.

Proof As $X_1 \sim \text{Bernoulli}\left(\frac{p(\beta^2+1)}{1+p\beta^2}\right)$, its mgf is expressed as $M_{X_1}(t) = \frac{1-p+pe^t(\beta^2+1)}{1+p\beta^2}$, similarly X_2 has the mgf $M_{X_2}(t) = (1-p+pe^t)^{\beta-1}$, respectively. Let $Y = X_1 + X_2$ the mgf of Y is

$$M_Y(t) = \frac{1-p+pe^t(\beta^2+1)}{1+p\beta^2} (1-p+pe^t)^{\beta-1},$$

that is,

$$M_Y(t) = M_{X_1}(t)M_{X_2}(t).$$

Proposition 2: If $Y_i \sim \mathcal{WBL}(p, \beta)$ for $i = 1, 2, 3, \dots, n$, are independent random variables, then $Z = Y_1 + Y_2 + Y_3 \dots + Y_n$, also has the $\mathcal{WBL}(n, p, \beta)$ distribution.

Proof

As $Y_i \sim \mathcal{WBL}(p, \beta)$ for $i = 1, 2, 3, \dots, n$, the mgf of $Z = Y_1 + Y_2 + Y_3 \dots + Y_n$ is expressed as

$$M_Z(t) = \prod_{i=1}^n M_{Y_i}(t) = \left(\frac{1-p+pe^t(\beta^2+1)}{1+p\beta^2} \right)^n (1-p+pe^t)^{n(\beta-1)}.$$

It can also be rewritten as

$$M_Y(t) = M_{X_{B1}}(t)M_{X_{B2}}(t),$$

where $X_{B1} \sim \text{Binomial}\left(n, \frac{p(\beta^2+1)}{1+p\beta^2}\right)$ and $X_{B2} \sim \text{Binomial}(n(\beta-1), p)$.

3. Parameter Estimation and Inference

In this section we have studied different methods of estimation namely moment method (MME)

and maximum likelihood (MLE) for $\mathcal{WBL}$ parameters.

3.1. Maximum Likelihood and Method of Moments Estimation

Maximum Likelihood Method (MLE): If $Y_1, Y_2, \dots, Y_n$ be a random sample drawn identically independently from the $\mathcal{WBL}$ distribution with observed values $x_1, x_2, \dots, x_n$, then their joint probability function as log-likelihood function is

$$\kappa_{p,\beta} = \ln(L(p; \beta)) = -n \ln(1 + \beta^2 p) + \sum_{i=1}^n \ln\left(\frac{\beta}{x_i}\right) + \sum_{i=1}^n \ln(1 + \beta x_i) + \ln p \sum_{i=1}^n x_i + \ln(1 - p) \sum_{i=1}^n (\beta - x_i).$$

Now on partially differentiating both sides of the above equation with respect to p and β and equating them to zero we get MLEs of p and β , respectively, as

(11)

$$\sum_{i=1}^n \frac{x_i}{p} + \sum_{i=1}^n \frac{\beta - x_i}{1 - p} = n\beta^2(1 + \beta^2 p),$$

And

(12)

$$\sum_{i=1}^n \frac{x_i}{1 + \beta x_i} + \sum_{i=1}^n \{\psi^{(0)}(1 + \beta) + \psi^{(0)}(1 + \beta - x_i)\} = 2n\beta p(1 + \beta^2 p)^{-1} - n \ln(1 - p).$$

Similarly, second derivative of equation (11) and equation(12) with respect to p and β respectively are

$$\begin{aligned} \frac{\partial^2 \kappa_{p,\beta}}{\partial p^2} &= n\beta^4(1 + \beta^2 p)^{-2} - \frac{\sum_{i=1}^n (\beta - x_i)}{(1 - p)^2} - \sum_{i=1}^n \frac{x_i}{p^2}, \\ \frac{\partial^2 \kappa_{p,\beta}}{\partial \beta^2} &= 4n\beta^2 p^2(1 + \beta^2 p)^{-2} - 2np(1 + \beta^2 p)^{-1} \\ &\quad + \sum_{i=1}^n \{\psi^{(1)}(1 + \beta) - \psi^{(1)}(1 + \beta - x_i)\} - \sum_{i=1}^n \frac{x_i^2}{(1 + \beta x_i)^2}, \end{aligned}$$

also second derivative of equation (11) with respect to β , or equation (12) with respect to p , yields

$$\frac{\partial^2 \kappa_{p,\beta}}{\partial \beta \partial p} = 2n\beta^3 p(1 + \beta^2 p)^{-2} - 2n\beta(1 + \beta^2 p)^{-1} - \frac{n}{1 - p}.$$

where $\psi^{(n)}(z) = \frac{d^n \psi(z)}{dz^n}$ is the logarithmic derivative of the gamma function. The MLEs are computed using computational package such as Mathematica [7.0]. In view of the regularity conditions (see Rohatgi and Saleh, 2002, pp. 419), the MLEs $\hat{p}$ and $\hat{\beta}$ of the $\mathcal{WBL}$ distribution has a bivariate normal distribution with mean (p, β) and variance-covariance matrix $(I(p, \beta))^{-1}$, where $I(p, \beta)$ denotes the information matrix and given by

$$I((p, \beta)|_{p=\hat{p}, \beta=\hat{\beta}}) = \begin{bmatrix} \mathbb{E}\left(\frac{-\partial^2 \kappa_{p,\beta}}{\partial p^2}\right) & \mathbb{E}\left(\frac{-\partial^2 \kappa_{p,\beta}}{\partial p \partial \beta}\right) \\ \mathbb{E}\left(\frac{-\partial^2 \kappa_{p,\beta}}{\partial p \partial \beta}\right) & \mathbb{E}\left(\frac{-\partial^2 \kappa_{p,\beta}}{\partial \beta^2}\right) \end{bmatrix}.$$

Moment Method of Estimation (MM): To apply the method of moments for estimating p and β , the population moments must be equated to the corresponding sample moments, and we must then solve the two equations simultaneously for p and β . Since the population moments form are not in closed form, then the moments estimators (MEs) of the $\mathcal{WBL}$ distribution can be obtained by equating the first two sample moments with the population moments, i.e.,

$$\frac{\beta p(1 + \beta^2 p + \beta(1 - p))}{1 + \beta^2 p} = \frac{\sum_{i=1}^n x_i}{n},$$

$$\frac{(\beta p(1 + \beta + (\beta - 1)(1 + 3\beta)p + (\beta - 2)(\beta - 1)\beta p^2))}{1 + \beta^2 p} = \frac{\sum_{i=1}^n x_i^2}{n}.$$

Alternatively, the moment estimates can also be obtained on using the method suggested by Khan et al. (1989). According to this method, β and p are estimated by minimizing

$$\left(\frac{\beta p(1 + \beta^2 p + \beta(1 - p))}{1 + \beta^2 p} - \frac{\sum_{i=1}^n x_i}{n} \right)^2$$

$$+ \left(\frac{(\beta p(1 + \beta + (\beta - 1)(1 + 3\beta)p + (\beta - 2)(\beta - 1)\beta p^2))}{1 + \beta^2 p} - \frac{\sum_{i=1}^n x_i^2}{n} \right)^2,$$

with respect to β and p . Since a closed form solution of this method can not be obtained so a numerical solution via computational package Mathematica [7.0] is adopted to find the estimators, see Table 2.

3.2. Simulation study

Here, some simulation studies with 1000 replications are carried out to compare the performance of different estimation procedures discussed in the last section. A general form to generate a random variable Y from the $\mathcal{WBL}(p, \beta)$ distribution is adopted by using the Mathematica [7.0] computational package. Estimators qualities are checked via two tools namely bias and mean square error (MSE). In practice, it is not possible to get unbiased estimator, and biased estimators are frequently used, generally the estimators with small bias are preferred. A biased estimator may be used for various reasons like, an unbiased estimator does not exist without further assumptions about a population or is difficult to compute; an estimator is median-unbiased but not mean-unbiased (or the reverse); a biased estimator reduces some loss function (particularly mean squared error) compared with unbiased estimators (notably in shrinkage estimators); or in some cases being unbiased is too strong a condition, and the only unbiased estimators are not useful. On the other hand MSE is a measure of the quality of an estimator; it is always non-negative, and values closer to zero are better. For this purpose we defined the two tools as

- Average bias of the simulated estimates:

$$\frac{1}{1000} \sum_{i=1}^{1000} (\Theta^* - \Theta),$$

where $\Theta^* = (\hat{p}, \hat{\beta})$ are the MLEs of the $\mathcal{WBL}(p, \beta)$ distribution .

- Average mean square error (MSE) of the simulated estimates:

$$\frac{1}{1000} \sum_{i=1}^{1000} (\Theta^* - \Theta)^2.$$

Table-1 concerns the bias and MSE of the MLEs for the $\mathcal{WBL}$ distribution. It shows the average bias and average MSE of the estimates for different values of p, β . A downward bias for $\hat{p}$ and $\hat{\beta}$ is observed for $p > 0.10$ and $\beta > 0$, however an upward bias for $\hat{p}$ is observed when $p < 0.1$ and $\beta < 1$, similar results are also observed in MSE of the estimates. However, the bias and MSE reduces as sample size increases.

Table-1: Average bias and MSE values of MLEs from simulation of the $\mathcal{WBL}$ distribution					
Parameter	Sample Size	Bias($\hat{p}$)	Bias($\hat{\beta}$)	MSE($\hat{p}$)	MSE($\hat{\beta}$)
$p = 0.0465, \beta = 0.5846$	$n = 25$	0.5183	-0.4658	0.2935	0.2263
	$n = 50$	0.3608	-0.3917	0.1352	0.1610
	$n = 150$	0.3478	-0.3907	0.1268	0.1606
	$n = 250$	0.2903	-0.3876	0.0857	0.1503
$p = 0.5986, \beta = 6.7868$	$n = 25$	-0.1507	-2.8397	0.0331	10.0326
	$n = 50$	-0.1426	-1.1234	0.0301	1.4917
	$n = 150$	-0.1085	-1.1218	0.0117	1.4862
	$n = 250$	-0.0970	-1.0923	0.0113	1.1542
$p = 0.7632, \beta = 10.4321$	$n = 25$	-0.1984	-1.1551	0.0393	1.3344
	$n = 50$	-0.1882	-0.9081	0.0366	0.9796
	$n = 150$	-0.1641	-0.8959	0.0274	0.87389
	$n = 250$	-0.1586	-0.0731	0.0180	0.0692
$p = 0.9527, \beta = 20.1563$	$n = 25$	-0.0421	-0.6709	0.0016	0.4502
	$n = 50$	-0.0411	-0.6077	0.0366	0.3693
	$n = 150$	-0.0315	-0.5014	0.0011	0.2514
	$n = 250$	-0.0301	-0.3064	0.0010	0.1045

Table-2: Average bias and MSE values of MMEs from simulation of the WBL distribution

Parameter	Sample Size	Bias($\hat{p}$)	Bias($\hat{\beta}$)	MSE($\hat{p}$)	MSE($\hat{\beta}$)
$p = 0.0465, \beta = 0.5846$	$n = 25$	0.0147	0.4154	0.0015	0.1725
	$n = 50$	0.0009	0.4154	0.0003	0.1725
	$n = 150$	-0.0026	0.4154	0.0001	0.1725
	$n = 250$	-0.0387	0.4154	0.0001	0.1725
$p = 0.5986, \beta = 6.7868$	$n = 25$	0.1472	-2.5606	0.0266	6.7035
	$n = 50$	0.1458	-2.5605	0.0261	6.6905
	$n = 150$	0.1371	-2.5604	0.0224	6.6437
	$n = 250$	0.1343	-2.5131	0.0183	6.6312
$p = 0.7632, \beta = 10.4321$	$n = 25$	0.0162	-2.9511	0.0071	9.0386
	$n = 50$	0.0154	-2.8397	0.0031	9.0223
	$n = 150$	0.0150	-2.8097	0.0029	9.0121
	$n = 250$	0.0122	-2.7885	0.0021	9.0024
$p = 0.9527, \beta = 20.1563$	$n = 25$	0.0152	-1.2844	0.0003	1.6677
	$n = 50$	0.0151	-1.2150	0.0003	1.4927
	$n = 150$	0.0109	-1.1367	0.0002	1.2963
	$n = 250$	0.0107	-1.1367	0.0001	1.2931

From Table-2, the MSE of the MM estimators tend to zero for large n and also, a downward bias is reflected in MM estimator of β when both p and β are less than 1. However, for $0.50 < p < 0.80$ and $5 < \beta < 15$, the MSE becomes largest for MMEs for small sample sizes.

4. Evaluation Statistics and Data Applications

4.2.1. Test and evaluation statistics:

In this section, we present some data sets to examine the fitting of the proposed model. Maximum likelihood estimation is used to obtain the parameter estimates of the compared distributions, for this purpose seed values obtain from method of moments are used, this can be done using software such as Mathematica[7.0]. Many of discrete statistical distributions exist in statistical literature, some of them are suitable in over-dispersion/under-dispersion scenario, others are appropriate for equidispersion and some are for all features. In this context, we studied those distributions which are very strong in their respective area, Poisson Lindley (PL) (over-dispersion), discrete version of gamma

($GAMMA$)(over-, under- and equi-dispersed), Binomial (BD)(under-dispersion) and Discrete Lindley

(DL) (over-, under- and equi-dispersions) distributions. The probability functions of the competing models are expressed in the following Table.

Table 3 The compared distributions to the WBL distribution

Name	The distribution formula	Reference	Domain
$\mathcal{PL}$	$\mathbb{P}(Y = x) = \frac{\beta^2(x + \beta + 2)}{(\beta + 1)^{x+3}}$	Sankaran(1970)	$x = 0, 1, 2, \dots$ and $\beta > 0$
$\mathcal{DL}$	$\mathbb{P}(Y = x) = \frac{(1-p)^2(1+\beta x)p^x}{(1+p(\beta-1))}$	Hussain et al., (2016)	$x = 0, 1, 2, \dots$ and $\beta > 0$
$\mathcal{BD}$	$\mathbb{P}(Y = x) = \binom{\beta}{x} p^x (1-p)^{\beta-x}$	Rohatgi et al.,(2002)	$x = 0, 1, 2, \dots, 0 < p < 1$ and $\beta > 0$
$\mathcal{GAMMA}$	$\mathbb{P}(Y = x) = \frac{\Gamma(p, \frac{x}{\beta}) - \Gamma(p, \frac{x+1}{\beta})}{\Gamma(p)}$	Chakraborty and Chakravarty(2012)	$x = 0, 1, 2, \dots, p > 0$ and $\beta > 0$

4.1. Evaluation tests

In order to demonstrate the proposed methodology, we consider two real life data sets and compared the above discussed probability models via the Akaike Information Criterion (AIC) and the Bayesian Information criterion (BIC) which are used to select the best model among several models. The definitions of AIC and BIC are:

$$AIC = 2k - 2\ell(\hat{\theta}), BIC = -2\ell(\hat{\theta}) + k\ln(\ln(n)).$$

Moreover, perfection of competing models is also tested via the Chi-Square (χ^2) test which is defined as $\chi^2 = \sum_{i=1}^m \frac{(o_i - e_i)^2}{e_i}$ with $(m - k - 1)$ degrees of freedom, where k denotes the number of parameters, n denotes the number of observations, m denotes the number of classes, and o_i and e_i are the observed and expected frequencies of the i^{th} class, respectively.

4.2. Real data examples with statistical analyses

In order to check the competency we consider two data sets. Their descriptions and our statistical analyses are given below.

Count Data Set: Fish catch data was reported at Table-6 by Kemp (1992, and later on by Deniz (2009). Clearly, the data set is under-dispersed because the sample variances are lower than the respective sample means.

Statistical analysis: Analysis of fish catch data shows that the proposed model is the only suitable model. As we can see in Table-4, the proposed model have not only the minimum values of the test statistics and higher p -value but also have least loss of information by showing least AIC and BIC .

Table 4: Observed and Expected Frequencies, Log Likelihood, χ^2 , AIC, BIC and p - values

x	Observed Frequency	WBL	DL	PL	BD	GAMMA
0	1	0.1	0.70	14.73	2.36	4.31
1	2	4.02	15.62	14.71	10.03	10.94
2	11	13.86	18.69	13.43	19.87	13.59
3	20	23.05	16.93	11.63	24.42	13.61
4	29	24.27	13.67	9.73	20.83	12.27
5	23	18.09	10.35	7.94	13.07	10.39
6	10	10.14	7.53	6.36	6.24	8.45
7	3	4.43	5.33	5.02	2.31	6.67
8	1	1.54	3.69	3.92	0.67	5.15
Total	100					
LL(-l)	—	181.92	210.48	236.46	190.65	218.12
χ^2	—	3.80	46.22	92.32	24.36	50.65
p - value		0.5786	0.0001	0.0001	0.0002	0.0001
AIC	—	367.84	424.96	474.92	385.3	440.24
BIC	—	373.05	430.17	477.53	390.51	445.45

Experimental Data Set: We consider the data concerning the Otis IQ Scores for 52 non-White males hired by a large insurance company in 1971 which was first reported by Roberts (1988) and later on by Sharafi and Behboodian (2008). The data set is given as 91, 102, 100, 117, 122, 115, 97, 109, 108, 104, 108, 118, 103, 123, 123, 103, 106, 102, 118, 100, 103, 107, 108, 107, 97, 95, 119, 102, 108, 103, 102, 112, 99, 116, 114, 102, 111, 104, 122, 103, 111, 101, 91, 99, 121, 97, 109, 106, 102, 104, 107, 95.

Statistical analysis: Analysis of data starts with the formation of classes by using the R computational package. The classes with related frequencies are displayed in the Table 6. As we can see in Table-5. The discrete Lindley and Poisson-Lindley distributions are completely unsatisfactory, so their analysis is not carried in Table —5. Further, the proposed model produces satisfactory results by yielding a higher p - value with least loss of information by showing least AIC and BIC.

Table-5: Observed and Expected Frequencies, Log Likelihood, χ^2 , AIC, BIC and p - values

Classes	Observed Frequency	WBL	BD	$GAMMA$
91-99	9	8.67	9.23	9.21
99-102	9	5.90	6.29	5.58
102-103	5	2.23	2.36	2.03
103-107	8	9.51	9.91	8.47
107-110	6	6.89	6.97	6.08
110-118	7	12.44	11.76	11.77
118-123	8	3.05	2.53	3.61
Total	52			
$LL(-l)$	—	183.35	183.42	184.69
χ^2	—	1.91	3.86	2.72
p - value		0.7523	0.4253	0.6057
AIC	—	370.70	370.84	373.38
BIC	—	374.60	374.74	377.28

5. CONCLUSION

We introduce a two parameter discrete distribution, so-called the WBL distribution. The new distribution is much more flexible than the binomial, discrete Lindley distribution and could have increasing and unimodal hazard rates. We derive explicit algebraic formula for the r^{th} moment which holds in generality for any parameter values. The estimation procedures for the parameters of WBL distribution are also derived considering two estimation methods whose convergence rate is very high as compare to the others. The MME method yields global estimates where as MLE yields local estimates. Since it is not feasible to compare these methods theoretically, we have presented an extensive simulation study in order to identify the most efficient procedure. We observed that the MLE provides a large number of local estimates whereas MME yields global one so we used this global value as a seed for finding the MLE. Moreover, we have also studied various mathematical properties including moments, ascending and descending factorial moments, negative and negative factorial moments, discrete self-decomposability, and infinite divisibility. Finally, two data-sets were analyzed for illustrative purposes

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